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An Analytic Approximation of Solutions of Stochastic Differential Equations

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Abstract—This paper is devoted to the construction of an approximate solution of the stochastic differential equation of the Ito type, defined on a partition of the time-interval. The coefficients of the equation by their Taylor series up to arbitrary derivatives are approximated. The closeness of the original and approximate solutions is measured in the sense of the L^p-norm and with probability one.

Keywords—Stochastic differential equation, Taylor approximation, L^p-convergence, Approximate solution, Convergence with probability one.

1. INTRODUCTION

Real phenomena in different fields of science and engineering, especially in financial consideration, involving stochastic excitations of a Gaussian white noise type have been extensively investigated both theoretically and experimentally over a long period of time. Remember that a Gaussian white noise is a tolerable abstraction and is never a completely faithful representation of a physical noise source, at least if mathematically described as the formal derivative of a Brownian motion process. Thus, all such problems are essentially based on a stochastic differential equation of the Ito type [1]. Since these equations are not solvable explicitly in most cases, it is important to find their approximate solutions in an explicit form, or in a form suitable for applications of numerical methods.

Further, all stochastic processes and random variables considered here are supposed to be defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We should mention that we shall restrict ourselves to the scalar case in this paper. For applications, extension to multidimensional case is of great importance and it is not difficult in itself, but is rather complicated in details and involves complex notations. Likewise, we usually restrict ourselves on the time interval $[0,1]$ instead of $[t_0, T]$.

In the present paper, we consider the stochastic differential equation of the Ito type,

$$dx_t = a(t, x_t)\, dt + b(t, x_t)\, dw_t, \qquad t \in [0,1], \quad x_0 = \eta. \tag{1}$$

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Here, $w = (w_t,\ t \geq 0)$ is a standard Wiener process with a natural filtration $\{\mathcal{F}_t,\ t \geq 0\}$ of nondecreasing sub-σ-algebras of $\mathcal{F}$ ($\mathcal{F}_t = \sigma\{w_s,\ 0 \leq s \leq t\}$), the initial condition η is a random variable independent of w, the functions $a : [0,1] \times R \to R$, $b : [0,1] \times R \to R$ are assumed to be Borel measurable on their domains. The process $x = (x_t,\ t \in [0,1])$ is a strong solution of equation (1) in the following way: x is adapted to $\{\mathcal{F}_t,\ t \geq 0\}$, the Lebesgue and Ito integrals in the integral form of equation (1) are well defined (that is, $\int_0^1 |a(t,s)|\,dt < \infty$, $\int_0^1 |b(t,x)|^2\,dt < \infty$), $x(0) = \eta$ a.s., and equation (1) is satisfied, a.s., for all $t \in [0,1]$.

On the basis of the classical theory of stochastic differential equations of the Ito type, one can prove the basic existence and uniqueness theorem ([1–5], for example), based on Picard method of iterations. If the functions a and b are globally Lipschitzian and satisfy the usual linear growth condition in the second argument, i.e., if there exists a constant $L > 0$ such that, for all $(t,x), (t,y) \in [0,1] \times R$,

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq L|x-y|, \tag{2}$$

$$|a(t,x)|^2 + |b(t,x)|^2 \leq L^2\left(1 + |x|^2\right), \tag{3}$$

and if $E|\eta|^{2m} < \infty$ for any positive integer m, then there exists a unique, a.s., continuous strong solution x of equation (1), satisfying $E\{\sup_{t\in[0,1]} |x_t|^{2m}\} < \infty$. Moreover, by applying the procedure used in [4,5], one can prove that if $E|\eta|^p < \infty$ for any number $p > 0$, then $E\{\sup_{t\in[0,1]} |x_t|^p\} < \infty$.

As mentioned above, in general it is not possible to determine explicitly the solution of equation (1). To obtain an approximate solution, some applicable analytic or numerical methods are usually used. The present paper is devoted to an analytic approximation for the solution of equation (1). In the next section, starting from the papers, first of all [6,7] by Atalla, we formulate the problem and we give our main results. We compare in the L^p-norm, $p \geq 2$, the solution of equation (1) to its approximations, by the solutions of stochastic differential equations determined on partitions of the interval $[0,1]$, in which the drift and diffusion coefficients are Taylor approximations of the functions a and b. Moreover, we shall prove that under the same conditions, the sequence of the approximate solutions tends with probability one to the solution of equation (1). In the remainder of this section, an illustrative example is presented to justify our theoretical result. We give some useful conclusions and we point out possible applications of the previous considerations.

2. FORMULATION OF THE PROBLEM AND MAIN RESULTS

There are a great number of papers in which the solution of equation (1) is approximated on an arbitrary partition of the interval $[0,1]$.

$$0 = t_0 < t_1 < \cdots < t_n = 1, \qquad \delta_n = \max_{0\leq k\leq n-1}(t_{k+1} - t_k). \tag{4}$$

Thus, Maruyama [8] was one of the first to use and investigate the square convergence of the Euler method, analogous to the Euler polygonal line in the deterministic case. This result was improved upon in the paper [9] by Kanagawa. In paper [10] by Kanagawa, under some restrictive conditions, the solution x of equation (1) is approximated by the solutions x^n, $n \in N$ of the equations

$$\begin{aligned} dx_t^n &= a\left(t_k, x_{t_k}^n\right)\,dt + b\left(t_k, x_{t_k}^n\right)\,dw_t, \qquad t \in [t_k, t_{k+1}), \quad 0 \leq k \leq n-1, \\ x_0^n &= \eta, \end{aligned} \tag{5}$$

in the sense of the L^p-norm, $p \geq 2$. The rate of this approximation is $O(\delta_n^{p/2})$ when $n \to \infty$ and $\delta_n \to 0$. This result was obtained earlier in [2] for $p = 2$.

Essentially, the basic ideas of the present paper go back to papers [6,7] by Atalla. In paper [6], under the Lipschitz condition (2) and the linear growth condition (3) for the functions a and b, it was shown that the solutions x^n of equations (5) approximate the solution x of equation (1) in the sense of the L^p-norm, $p \geq 2$, with the rate of approximation $O(\delta_n^{p/2})$ when $n \to \infty$ and $\delta_n \to 0$. In fact, the solution $x^n = x_t^n$, $t \in [0,1]$ is constructed as an a.s. continuous process, by successively connecting the processes $(x_t^n,\ t \in [t_k, t_{k+1}])$ at the points t_k, $0 \leq k \leq n-1$ of partition (4). Later on, in paper [11], this concept was appropriately extended to stochastic integrodifferential equations of the Ito type.

In order to improve the previous result, Atalla approximated the solution of equation (1) by the solutions of stochastic differential equations determined on partition (4), whose drift and diffusion coefficients were Taylor approximations of the functions a and b, up to the first derivatives in argument x, i.e., by linear stochastic differential equations,

$$\begin{aligned} dx_t^n &= \left[a\left(t_k, x_{t_k}^n\right) + a_x'\left(t_k, x_{t_k}^n\right)\left(x_t^n - x_{t_k}^n\right)\right] dt \\ &+ \left[b\left(t_k, x_{t_k}^n\right) + b_x'\left(t_k, x_{t_k}^n\right)\left(x_t^n - x_{t_k}^n\right)\right] dw_t, \qquad t \in [t_k, t_{k+1}), \quad 0 \leq k \leq n-1, \quad x_0^n = \eta. \end{aligned} \tag{6}$$

The approximate solutions x^n are constructed as in the paper [6], by connecting the solutions of equations (6) at the partition points. The rate of the approximation in the L^p-th norm of the solutions x and x^n is $O(\delta_n^p)$ when $n \to \infty$ and $\delta_n \to 0$.

Following the previous concept and bearing in mind that the Taylor formula, as a polynomial, has proven to be a very useful tool in both theoretical and practical investigations, it is logical to ask: what is the rate of the approximation to the solution x by the approximate solution x^n, if the drift and diffusion coefficients are Taylor approximations of the functions a and b, up to arbitrary fixed derivatives? The main goal of the present paper is to answer this question.

Therefore, we shall investigate equation (1) in the equivalent integral form

$$x_t = \eta + \int_0^t a(s, x_s)\, ds + \int_s^t b(s, x_s)\, dw_s, \qquad t \in [0,1], \tag{7}$$

by approximating its solution $x = (x_s,\ s \in [0,t])$ on partition (4) with the solutions of the following equations:

$$\begin{aligned} x_t^n &= x_{t_k}^n + \int_{t_k}^t \sum_{i=1}^{m_1} \frac{a_x^{(i)}\left(s, x_{t_k}^n\right)}{i!}\left(x_s^n - x_{t_k}^n\right)^i ds \\ &+ \int_{t_k}^t \sum_{i=1}^{m_2} \frac{b_x^{(i)}\left(s, x_{t_k}^n\right)}{i!}\left(x_s^n - x_{t_k}^n\right)^i dw_s, \qquad 0 \leq k \leq n-1, \end{aligned} \tag{8}$$

where $x_{t_0} = \eta$, a.s. The approximate solution $x^n = (x_t^n,\ t \in [0,1])$ is constructed as above, as an a.s. continuous process, by successively connecting the processes $(x_t^n,\ t \in [t_k, t_{k+1}])$ at the points t_k, $0 \leq k \leq n-1$ of partition (4).

Of course, we must require the functions a and b to satisfy appropriate conditions, first of all, to be sufficiently smooth. We shall assume the existence and uniqueness of the approximate solutions for equations (8), without emphasizing appropriate conditions for the coefficients of these equations, and we shall emphasize only the conditions explicitly used in our discussion.

In addition to the above assumptions that the functions a and b satisfy the Lipschitz condition (2) and the linear growth condition (3), we introduce the following assumptions.

($\mathcal{A}_1$) The functions a and b have Taylor approximation in the second argument, up to m_1^{th} and m_2^{th} derivatives, respectively.

($\mathcal{A}_2$) The functions $a_x^{(m_1+1)}(t,x)$ and $b_x^{(m_2+1)}(t,x)$ are uniformly bounded, i.e., there exist positive constants L_1 and L_2, such that

$$\sup_{[0,1]\times R}\left|a_x^{(m_1+1)}(t,x)\right| \leq L_1, \qquad \sup_{[0,1]\times R}\left|b_x^{(m_2+1)}(t,x)\right| \leq L_2.$$

($\mathcal{A}_3$) There exist unique, a.s., continuous strong solutions x and x^n of equations (7) and (8), respectively, satisfying $E\{\sup_{[0,1]} |x_t|^p\} < \infty$ and $E\{\sup_{[0,1]} |x_t^n|^{(M+1)^2 p}\} \le Q < \infty$, where $M = \max\{m_1, m_2\}$ and Q is a positive constant. All Lebesgue and Ito integrals employed further are also well defined.

Clearly, because of Assumption ($\mathcal{A}_3$), it will be necessary to suppose $E|\eta|^{(M+1)^2 p} < \infty$, so that the limit Q is independent of n and δ_n.

First, let us prove the following assertion, which is of independent interest with respect to the previously mentioned problem, but which plays an important role in the description of the closeness between the solutions x and x^n in the sense of the L^p-norm.

PROPOSITION 1. *Let x^n be the solution of equation (8) and conditions (2), (3), ($\mathcal{A}_1$)–($\mathcal{A}_3$) be satisfied. Then, for $0 \le r \le (M+1)p$,*

$$E\left|x_t^n - x_{t_k}^n\right|^r \le C\delta_n^{r/2}, \qquad t \in [t_k, t_{k+1}], \quad k = \overline{0, n-1},$$

where C is a generic constant independent of n and δ_n.

PROOF. For notation simplicity reason, let us denote that

$$A\left(t, x_{t_k}^n, x_t^n\right) = \sum_{i=0}^{m_1} \frac{a_x^{(i)}\left(t, x_{t_k}^n\right)}{i!}\left(x_t^n - x_{t_k}^n\right)^i,$$
$$B\left(t, x_{t_k}^n, x_t^n\right) = \sum_{i=0}^{m_2} \frac{b_x^{(i)}\left(t, x_{t_k}^n\right)}{i!}\left(x_t^n - x_{t_k}^n\right)^i.$$

In order to estimate $E|x_t^n - x_{t_k}^n|^p$, we shall first apply the elementary inequality $|a+b|^r \le (2^{r-1} \vee 1)(|a|^r + |b|^r)$, $r \ge 0$, to equation (8), and after that Hölder inequality to Lebesgue integral, as well as Burkholder-Davis-Gundy inequality [2,4,5] to the Ito integral: for any $l > 0$, there exists a constant $c_l > 0$, such that $E\sup_{s\in[t_0,t]} |\int_{t_0}^s f_u\, dw_u|^l \le c_l E(\int_{t_0}^t |f_u|^2\, du)^{l/2}$ for any measurable $\mathcal{F}_t$-adapted process $(f_t,\ t \in [0,T])$, where $\int_{t_0}^T |f_t|^2\, dt < \infty$, a.s. In fact, we use this inequality in which the left-hand side is minorized by omitting supremum. So, for all $t \in [t_k, t_{k+1}]$, $0 \le k \le n-1$, we obtain

$$\begin{aligned}
E\left|x_t^n - x_{t_k}^n\right|^r &\le 2^{r-1}\left[E\left|\int_{t_k}^t A\left(s, x_{t_k}^n, x_s^n\right)\, ds\right|^r + E\left|\int_{t_k}^t B\left(s, x_{t_k}^n, x_s^n\right)\, dw_s\right|^r\right] \\
&\le 2^{r-1}\left[(t-t_k)^{r-1}\int_{t_k}^t E\left|A\left(s, x_{t_k}^n, x_s^n\right)\right|^r ds\right. \\
&\quad \left. + c_r(t-t_k)^{r/2-1}\int_{t_k}^t E\left|B\left(s, x_{t_k}^n, x_s^n\right)\right|^r ds\right] \\
&\equiv 2^{r-1}(t-t_k)^{r/2-1}\left[(t-t_k)^{r/2}J_1(t) + c_r J_2(t)\right].
\end{aligned}$$

To estimate $J_1(t)$ and $J_2(t)$, we shall use Assumptions (3), ($\mathcal{A}_1$)–($\mathcal{A}_3$). Thus, for $\theta \in (0,1)$ we find

$$\begin{aligned}
J_1(t) &= \int_{t_k}^t E\left|a\left(s, x_s^n\right) - \left[a\left(s, x_s^n\right) - A\left(s, x_{t_k}^n, x_s^n\right)\right]\right|^r ds \\
&\equiv \int_{t_k}^t E\left|a\left(s, x_s^n\right) - \frac{a_x^{(m_1+1)}\left(s, x_{t_k}^n + \theta\left(x_s^n - x_{t_k}^n\right)\right)}{(m_1+1)!}\left(x_s^n - x_{t_k}^n\right)^{m_1+1}\right|^r ds \\
&\le 2^{r-1}\int_{t_k}^t \left[E\left(\left|a\left(s, x_s^n\right)\right|^2\right)^{r/2} + \frac{L_1^r}{[(m_1+1)!]^r} E\left|x_s^n - x_{t_k}^n\right|^{(m_1+1)r}\right] ds
\end{aligned}$$

$$\leq 2^{r-1}\int_{t_k}^{t}\left[2^{r/2-1}L^r\left(1+E|x_s^n|^r\right)\right.$$
$$\left.+2^{(m_1+1)r-1}\frac{L_1^r}{[(m_1+1)!]^r}\left(E\left|x_s^n\right|^{(m_1+1)r}+E\left|x_{t_k}^n\right|^{(m_1+1)r}\right)\right]ds$$
$$\leq 2^{r-1}\left[2^{r/2-1}L^r(1+R)+2^{(m_1+1)r}\frac{L_1^r R}{[(m_1+1)!]^r}\right](t-t_k)$$
$$\equiv C_1(t-t_k),$$

where $R=Q+1$ and $C_1=C_1(L,L_1,R,r,m_1)$ is a generic constant.

Similarly, by repeating completely the previous procedure, we estimate $J_2(t)$,

$$J_2(t)\leq C_2(t-t_k),$$

where $C_2=C_2(L,L_2,R,r,m_2)$. Therefore,

$$E\left|x_t^n-x_{t_k}^n\right|^r\leq 2^{r-1}(t-t_k)^{r/2}\left[C_1(t-t_k)^{r/2}+c_rC_2\right]\leq C(t-t_k)^{r/2}\leq C\delta_n^{r/2},$$

where C is a generic constant, which finishes the proof. ∎

Note that Proposition 1 could be proved with more general assumptions instead of $(\mathcal{A}_2)$, that the functions $a_x^{(m_1+1)}(t,x)$ and $b_x^{(m_2+1)}(t,x)$ satisfy the linear growth conditions in argument x. In this case, the proof will be somewhat complicated.

The following assertion is closely connected with our main results and provides the estimation of the closeness between the solutions x and x^n.

PROPOSITION 2. *Let x and x^n be the solutions of equations (7) and (8), respectively, and conditions (2), (3), $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ be satisfied. Then, for $p\geq 2$,*

$$\sup_{t\in[0,1]}E\left|x_t-x_t^n\right|^p\leq H\delta_n^{(m+1)p/2},$$

where $m=\min\{m_1,m_2\}$ and H is a generic constant independent of n and δ_n.

PROOF. Let $p>2$ and $t\in[t_k,t_{k+1}]$. If we subtract equations (7) and (8), we obtain

$$x_t-x_t^n=x_{t_k}-x_{t_k}^n+\int_{t_k}^{t}\left[a(s,x_s)-A\left(s,x_{t_k}^n,x_s^n\right)\right]ds+\int_{t_k}^{t}\left[b(s,x_s)-B\left(s,x_{t_k}^n,x_s^n\right)\right]dw_s.$$

By applying Ito's differential formula to the function $f(x)=|x|^p$ and then by taking expectation, we get

$$E\left|x_t-x_t^n\right|^p\leq E\left|x_{t_k}-x_{t_k}^n\right|^p+pE\int_{t_k}^{t}\left[a(s,x_s)-A\left(s,x_{t_k}^n,x_s^n\right)\right]\left|x_s-x_s^n\right|^{p-1}ds$$
$$+\frac{p(p-1)}{2}E\int_{t_k}^{t}\left[b(s,x_s)-B\left(s,x_{t_k}^n,x_s^n\right)\right]^2\left|x_s-x_s^n\right|^{p-2}ds$$
$$+pE\int_{t_k}^{t}\left[b(s,x_s)-B\left(s,x_{t_k}^n,x_s^n\right)\right]\left|x_s-x_s^n\right|^{p-1}dw_s$$
$$\equiv E\left|x_{t_k}-x_{t_k}^n\right|^p+pI_1(t)+\frac{p(p-1)}{2}I_2(t)+pI_3(t).$$

Let us note that $\Delta_t=E|x_t-x_t^n|^p$. Since $I_3(t)\equiv 0$, it follows that

$$\Delta_t=\Delta_{t_k}+pI_1(t)+\frac{p(p-1)}{2}I_2(t),\qquad t\in[t_k,t_{k+1}],\quad k=\overline{0,n-1}.\tag{9}$$

To estimate $I_1(t)$, we shall apply the Lipschitz condition (2) and the elementary Young inequality, which states for every $a, b \geq 0$ and $p > 1$, $1/p + 1/q = 1$, it follows that $ab \leq a^p/p + b^q/q$. Then,

$$\begin{aligned}
I_1(t) &\leq E\int_{t_k}^{t} \left|a(s,x_s) - a\left(s,x_s^n\right) + a\left(s,x_s^n\right) - A\left(a, x_{t_k}^n, x_s^n\right)\right| |x_s - x_s^n|^{p-1}\, ds \\
&\leq E\int_{t_k}^{t} \left[|a(s,x_s) - a(s,x_s^n)| \right. \\
&\quad \left. + \frac{\left|a_x^{(m_1+1)}\left(s, x_{t_k}^n + \theta_1\left(x_s^n - x_{t_k}^n\right)\right)\right|}{(m_1+1)!} \left|x_s^n - x_{t_k}^n\right|^{m_1+1} \right] |x_s - x_s^n|^{p-1}\, ds \\
&\leq E\int_{t_k}^{t} \left[L|x_s - x_s^n| + \frac{L_1}{(m_1+1)!}\left|x_s^n - x_{t_k}^n\right|^{m_1+1} \right] |x_s - x_s^n|^{p-1}\, ds \\
&\leq L\int_{t_k}^{t} \Delta_s\, ds + \frac{L_1}{(m_1+1)!} E\int_{t_k}^{t} \left|x_s^n - x_{t_k}^n\right|^{m_1+1} |x_s - x_s^n|^{p-1}\, ds.
\end{aligned}$$

The application of Hölder inequality for $\nu = p$, $\mu = p/(p-1)$ to the second term, after that Young inequality and Proposition 1, gives us

$$\begin{aligned}
I_1(t) &\leq L\int_{t_k}^{t} \Delta_s\, ds + \frac{L_1}{(m_1+1)!}\int_{t_k}^{t} \left(E\left|x_s^n - x_{t_k}^n\right|^{(m_1+1)p}\right)^{1/p} \left(E|x_s - x_s^n|^p\right)^{(p-1)/p} ds \\
&\leq \left[L + \frac{L_1(p-1)}{p(m_1+1)!}\right]\int_{t_k}^{t} \Delta_s\, ds + \frac{L_1}{p(m_1+1)!} C_1 \delta_n^{(m_1+1)p/2}(t - t_k).
\end{aligned}$$

Similarly, by using the Hölder inequality for $\mu = p/2$ and $\nu = p/(p-2)$, we estimate

$$I_2(t) \leq \left[2L^2 + \frac{2(p-2)L_2^2}{[p(m_2+1)!]^2}\right]\int_{t_k}^{t} \Delta_s\, ds + \frac{4L_2^2}{p[(m_2+1)!]^2} C_2 \delta_n^{(m_2+1)p/2}(t - t_k).$$

Now, the preceding estimations for $I_1(t)$ and $I_2(t)$, together with (9), imply

$$\Delta_t \leq \Delta_{t_k} + \alpha\delta_n^{(m+1)p/2}(t - t_k) + \beta\int_{t_k}^{t} \Delta_s\, ds, \tag{10}$$

where α and β are generic constants independent of n and δ_n. An application of the well-known Gronwall-Belmann inequality [12] yields

$$\Delta_t \leq \left[\Delta_{t_k} + \alpha\delta_n^{(m+1)p/2}(t - t_k)\right] e^{\beta(t-t_k)}, \qquad t \in [t_k, t_{k+1}], \quad k = \overline{0, n-1}. \tag{11}$$

Further, to prove our assertion, we apply the procedure earlier used in [7,11]. Since $x_0 = x_0^n = \eta$ and by taking $t = t_{k+1}$ in (11), we come to the following recurrence formula:

$$\begin{aligned}
&\Delta_{t_0} = 0, \\
&\Delta_{t_{k+1}} \leq \left[\Delta_{t_k} + \alpha\delta_n^{(m+1)p/2}(t_{k+1} - t_k)\right] e^{\beta(t_{k+1}-t_k)}, \qquad k = \overline{0, n-1}.
\end{aligned}$$

So, we easily find

$$\Delta_{t_k} \leq \alpha\delta_n^{(m+1)p/2}\sum_{i=0}^{k-1}(t_{i+1} - t_i)e^{\beta(t_k - t_i)} \leq \alpha e^{\beta}\delta_n^{(m+1)p/2}, \qquad k = \overline{1, n}.$$

By taking this estimation for Δ_{t_k} on the right side of (11), we get

$$\Delta_t \le H\delta_n^{(m+1)p/2}, \qquad t\in[t_k,t_{k+1}], \quad k=\overline{1,n-1},$$

where H is a generic constant. Thus, $\sup_{t\in[0,1]}\Delta_t \le H\delta_n^{(m+1)p/2}$.

For $p=2$, from

$$E\left|x_t-x_t^n\right|^2 \le 3E\left[\left|x_{t_k}-x_{t_k}^n\right|^2+\left|\int_{t_k}^t \left[a(s,x_s)-A\left(s,x_{t_k}^n,x_s^n\right)\right]\,ds\right|^2 \right.$$
$$\left. +\left|\int_{t_k}^t \left[b(s,x_s)-B\left(s,x_{t_k}^n,x_s^n\right)\right]\,dw_s\right|^2\right],$$

by applying the usual Ito integral isometry and Proposition 1, we easily deduce that relation (10) is also valid. Thus, this assertion is completely proved. ∎

In view of the preceding assertion, we can expect that the sequence of the approximate solutions $\{x^n,\ n\in N\}$ tends to the solution x as $n\to\infty$ and $\delta_n\to 0$, in the sense of the L^p-norm. This assertion immediately follows from the next theorem, which is the main result of this paper.

THEOREM 1. *Let the conditions of Proposition 2 be satisfied. Then, for $p\ge 2$,*

$$E\left\{\sup_{t\in[0,1]}\left|x_t-x_t^n\right|^p\right\} = O\left(\delta_n^{(m+1)p/2}\right), \qquad \text{when } n\to\infty, \quad \delta_n\to 0.$$

PROOF. To prove this assertion, we shall use the previous treatments, without especially emphasizing any step. Thus,

$$\begin{aligned}E\left\{\sup_{t\in[0,1]}\left|x_t-x_t^n\right|^p\right\} &\le 2^{p-1}\left[E\left\{\sup_{t\in[0,1]}\left|\int_0^t \left[a(s,x_s)-A\left(s,x_{t_k}^n,x_s^n\right)\right]\,ds\right|^p\right\}\right.\\ &\quad\left.+E\left\{\sup_{t\in[0,1]}\left|\int_0^t \left[b(s,x_s)-B\left(s,x_{t_k}^n,x_s^n\right)\right]\,dw_s\right|^p\right\}\right]\\ &\le 2^{p-1}\left[E\left|\int_0^1 \left[a(s,x_s)-A\left(s,x_{t_k}^n,x_s^n\right)\right]\,ds\right|^p\right.\\ &\quad\left.+c_pE\left|\int_0^1 \left[b(s,x_s)-B\left(s,x_{t_k}^n,x_s^n\right)\right]\,ds\right|^p\right]\\ &\le 2^{p-1}\left[\int_0^1 E\left|a(s,x_s)-A\left(s,x_{t_k}^n,x_s^n\right)\right|^p\,ds\right.\\ &\quad\left.+c_p\int_0^1 E\left|b(s,x_s)-B\left(s,x_{t_k}^n,x_s^n\right)\right|^p\,ds\right]\\ &\equiv 2^{p-1}[S_1+c_pS_2].\end{aligned} \tag{12}$$

By repeating the procedure used in the proof of Proposition 1 and by applying Proposition 2, we easily find

$$S_1 \le 2^{p-1}L^p\left[H+\frac{C_1}{[(m_1+1)!]^p}\right]\delta_n^{(m_1+1)p/2},$$

and analogously for S_2. Thus, from (12) we conclude that

$$\begin{aligned}E\left\{\sup_{t\in[0,1]}\left|x_t-x_t^n\right|^p\right\} &\le 4^{p-1}L^p\left[2H+\frac{C_1}{[(m+1)!]^p}+\frac{c_pC_2}{[(m_2+1)!]^p}\right]\delta_n^{(m+1)p/2},\\ &= O\left(\delta_n^{(m+1)p/2}\right), \qquad \text{when } n\to\infty, \quad \delta_n\to 0,\end{aligned} \tag{13}$$

which establishes the assertion. ∎

Therefore, $x^n \xrightarrow{L^p} x$, when $n\to\infty$ and $\delta_n\to 0$.

Moreover, another important result can be proved, the convergence with probability one of the previously considered sequence of approximations to the solution of equation (7).

THEOREM 2. *Under the conditions of Theorem 1, for $\delta_n = 1/n$, the sequence $(x^n,\, n \in N)$ of the solutions of equations (8) converges with probability one to the solution x of equation (7).*

PROOF. From (13), it follows that

$$E\left\{\sup_{t\in[0,1]} |x_t - x_t^n|^p\right\} \le K\delta_n^{(m+1)p/2},$$

where K is a constant independent of n and δ_n. Since $\delta_n = 1/n$, by applying the Chebyshev inequality we find, for arbitrary $\eta > 0$,

$$\sum_{n=1}^{\infty} P\left\{\sup_{t\in[0,1]} |x_t - x_t^n|^{p/2} \ge \frac{1}{n^\eta}\right\} \le \sum_{n=1}^{\infty} n^{2\eta} E\left\{\sup_{t\in[0,1]} |x_t - x_t^n|^p\right\} \le K \sum_{n=1}^{\infty} \frac{1}{n^{(m+1)p/2-2\eta}}.$$

The series on the right side is convergent for $p = 2$, $\eta = 1/3$, and $m \ge 1$, or for $p > 2$, $\eta < (p/2-1)/2$, and $m \ge 0$. Therefore, in the first case, and analogously in the second one, by applying the Borel-Cantelli lemma, it follows that

$$P\left\{\sup_{t\in[0,1]} |x_t - x_t^n| \ge \frac{1}{n^{1/3}} \text{ infinitely often}\right\} = 0,$$

i.e., for all large n,

$$\sup_{t\in[0,1]} |x_t - x_t^n| < \frac{1}{n^{1/3}} \text{ with probability one.}$$

Therefore, the sequence $(x^n,\, n \in N)$ converges with probability one to the solution x, uniformly on $[0,1]$. Thus, the proof becomes complete. ∎

EXAMPLE. In the remainder, we shall examine the validity of the preceding concept and results, by applying them to the following autonomous stochastic differential equation:

$$dx_t = (5x_t - 6\sin x_t)\,dt + (2\cos x_t - 1)\,dw_t, \qquad t \in [0,1], \quad x_0 = 0, \tag{14}$$

which is not reducible and explicitly solvable. Note that the drift and diffusion coefficients $a(x) = 5x - 6\sin x$ and $b(x) = 2\cos x - 1$ satisfy the conditions of Theorem 1, they are Lipschitzian, and satisfy the linear growth condition and Conditions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$.

It seems more convenient to transform the subintervals $[t_k, t_{k+1}]$ of partition (4) into $[0, t_{k+1} - t_k]$, and after that form the approximate equations (8) by using Maclaurin approximations of the functions $a(x)$ and $b(x)$, instead of their Taylor approximations near the points x_{t_k}. In fact, it means to introduce the following time translation:

$$t = t_k + u,$$

for which $[t_k, t_{k+1}] \to [0, t_{k+1} - t_k]$, $k = \overline{0, n-1}$. Then, we have a new Wiener process $\tilde{w}$ and a new unknown process $\tilde{x}$, instead of w and x, such that

$$\tilde{w}_u = w_{t_k+u}, \text{ a.s.}, \qquad \tilde{x}_u = x_{t_k+u}, \text{ a.s.} \tag{15}$$

Likewise, equation (14) becomes

$$\begin{aligned} d\tilde{x}_u &= (5\tilde{x}_u - 6\sin\tilde{x}_u)\,du + (2\cos\tilde{x}_u - 1)\,d\tilde{w}_u, \qquad u \in [0, t_{k+1} - t_k], \\ k &= \overline{0, n-1}. \end{aligned} \tag{16}$$

The initial condition is $\tilde{x}_0 = 0$, a.s., for $k = 0$; the initial conditions for $k = \overline{1, n-1}$ are successively obtained as the values of the process $\tilde{x}_u$ in the right side of the intervals $[0, t_k - t_{k-1}]$.

Maclaurin approximations of the functions $a(x)$ and $b(x)$, up to the third and second derivatives, are

$$a(x) \approx x^3 - x, \qquad b(x) \approx 1 - x^2,$$

such that the approximate solution $(\tilde{x}_u^n,\ u \in [0,1])$ is constructed by using successively the solutions of the equations

$$\begin{aligned} d\tilde{x}_u^n &= \left[(\tilde{x}_u^n)^3 - \tilde{x}_u^n\right] du + \left[1 - (\tilde{x}_u^n)^2\right] d\tilde{w}_u, \qquad u \in [0, t_{k+1} - t_k], \\ k &= \overline{0, n-1}. \end{aligned} \tag{17}$$

Clearly, the initial condition is $\tilde{x}_0^n = 0$, a.s., for $k = 0$, and for $k = \overline{1, n-1}$ they are successively obtained as the values of the process $\tilde{x}_u^n$ in the right side of the preceding intervals $[0, t_k - t_{k-1}]$.

Equations (17) are explicitly solvable (see [5, p. 122], for example) and have the solutions

$$\tilde{x}_u^n = \tanh\left(\tilde{w}_u + \operatorname{arctanh} \tilde{x}_0^n\right), \qquad u \in [0, t_{k+1} - t_k], \quad k = \overline{0, n-1}.$$

Having in mind (15), it follows that

$$x_{t_k+u}^n = \tanh\left(w_{t-t_k} + \operatorname{arctanh} x_{t_k}^n\right),$$

and, therefore,

$$x_t^n = \tanh\left(w_t - w_{t_k} + \operatorname{arctanh} x_{t_k}^n\right), \qquad t \in [t_k, t_{k+1}], \quad k = \overline{0, n-1}. \tag{18}$$

Thus, the solution $x^n = (x_t^n,\ t \in [0,1])$ is obtained by connecting processes $(x_t^n,\ t \in [t_k, t_{k+1}])$, $k = \overline{0, n-1}$, in the partition points. The rate of the closeness in the L^p-norm is

$$E\left\{\sup_{t\in[0,1]} |x_t - x_t^n|^p\right\} \le K\delta_n^{3p/2},$$

where K is a constant, obtained from (13). Likewise, for $\delta_n = 1/n$, by applying Theorem 2 it follows that $x^n \to x$ as $n \to \infty$, with probability one.

Note that solutions (18) can be also expressed in an equivalent form,

$$x_t^n = \frac{\left(1 + x_{t_k}^n\right) e^{2\left(w_t - w_{t_k}\right)} - \left(1 - x_{t_k}^n\right)}{\left(1 + x_{t_k}^n\right) e^{2\left(w_t - w_{t_k}\right)} + \left(1 - x_{t_k}^n\right)}. \tag{19}$$

By applying the previously described procedure, in Figure 1 the trajectory of the approximate solution on the partition of 30 uniformly ordered points is constructed, in which the Wiener process is generated by the standard Polar Marsaglia method.

Since $\delta_{30} = 1/30$, the rate of the closeness between the solution x_t of equation (14) and the approximate solution x_t^{30} expressed by (19), from (13), is

$$E\left\{\sup_{t\in[0,1]} \left|x_t - x_t^{30}\right|^p\right\} \le O\left(30^{-3p/2}\right).$$

Especially, for $p = 2$ and $p = 4$, the closeness is of the size 10^{-5} and 10^{-9}, respectively. ∎

Let us make some comments.

The analytic method presented in this paper is effectively applicable if the approximate equation can be explicitly solved. If not, because polynomials are very useful functional forms, the approximations obtained in this paper may also be useful in other applications of stochastic Taylor expansion, especially in the construction of various time discrete approximations of Ito

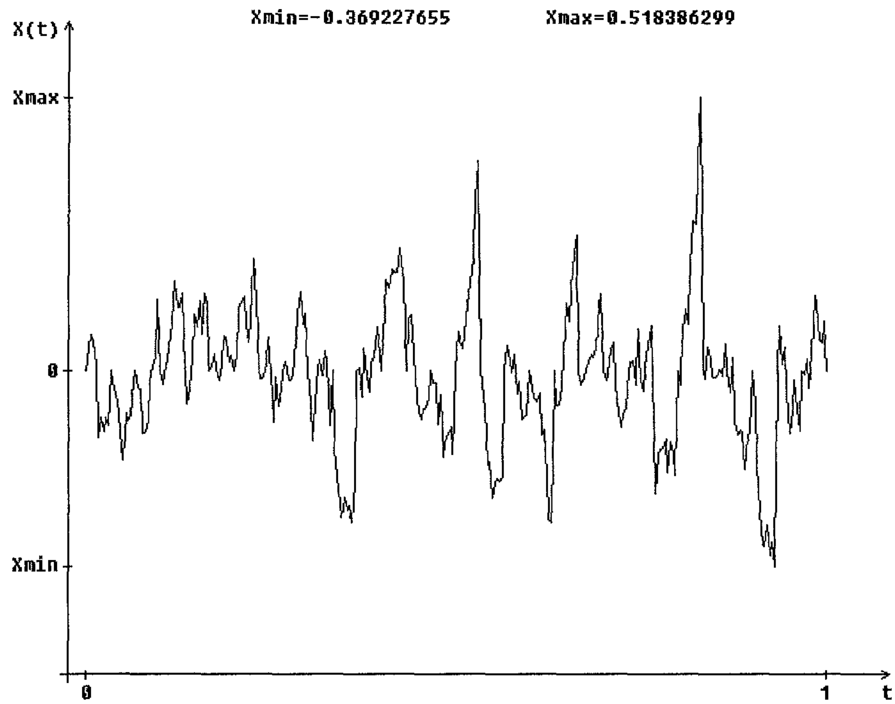

Figure 1. The trajectory of the approximate solution.

processes by using the Ito-Taylor expansion, which is the key to stochastic numerical analysis (see [5,13], for example).

The method exposed in this paper could be appropriately extended to stochastic integral and integrodifferential equations of the Ito type, as well as stochastic differential equations including martingales and martingale measures instead of the Brownian motion process.

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